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On quantization of singular varieties and applications to D-branes

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Abstract: We calculate the ring of differential operators on some singular affine varieties (intersecting stacks, a point on a singular curve or an orbifold). Our results support the proposed connection of the ring of differential operators with geometry of D-branes in (bosonic) string theory. In particular, the answer does know about the resolution of singularities in accordance with the string theory predictions.

Keywords: Bosonic Strings, D-branes, Differential and Algebraic Geometry.

Contents

1. Introduction

Merkulov [1] proposed a construction for deformation quantization of affine varieties. In particular, he considered quantization of the n-tuple point $x^n = 0$ and has proved the quantum algebra of functions on the associated phase space to be the matrix algebra $\mathrm{Mat}(n)$. Quantization of n coincident hyperplanes in $\mathbb{R}^{N+1}$ gives the tensor product of the matrix algebra $\mathrm{Mat}(n)$ and N copies of the Heisenberg algebra related to directions along the hyperplanes. This result reminds the appearance of the non-abelian degrees of freedom on the stack of n coincident D-branes [2, 3]. Recently the relevance of this procedure to D-brane physics (in particular, to boundary string field theory [4]–[7]) was established quantitatively in [8].

The present note aims to apply this technique to some singular varieties and to compare the result to that known from D-brane physics.

The paper is organized as follows. In section 2 we describe the procedure of quantization of affine varieties, including singular ones. The methods of algebraic geometry allow us to perform the quantization. This is given in terms of the ring of differential operators on a subvariety. We propose the explicit description of this ring in terms of an arbitrary resolvent of the ring of functions on the subvariety. According to the proposed connection with the geometry of D-branes, the properties of the ring of differential operators should capture, in particular, the unbroken gauge symmetry in the open string sector. In section 3 we study several examples and compare the results with the predictions of string theory. In the case of intersecting stacks we find a good agreement with physics: two subalgebrae responsible for the non-abelian degrees of freedom living on the worldvolume of each D-brane and non-local operators (massive modes of the strings stretching between branes). The structure of the resulting algebra of differential operators appears especially transparent in terms of the Meier-Vietoris sequence. For a point on a singular curve ("cusp") the quantum algebra behaves just as if the singularity was blown up (resolved). In the case of the point on the $\mathbb{C}^2/\mathbb{Z}_m$ orbifold we find a perfect agreement with the string theory picture: when the point approaches the singularity, the dimension of the algebra increases m^2 times. The same result is obtained using the blow-up.

Thus we found the complete agreement with bosonic string theory picture. It would be interesting to generalize these considerations to the case of superstrings. This will be done elsewhere.

2. Reduction of the ring of differential operators onto a subvariety

Algebraic geometry studies geometric concepts using the technique of (commutative) algebra. An important motivation for such a study is application to reducible or singular varieties (e.g. [9]). The ring of polynomial functions on an (affine algebraic) variety M embedded into the affine space $\mathbb{A}^N$ is given as the factor of the ring of polynomials in $x^1,\ldots,x^N$ w.r.t. the ideal generated by the set of equations defining M:

$$k[M] = k[x^1,\ldots,x^N]/(\phi_1,\ldots,\phi_n)\,, \tag{2.1}$$

$\phi_\alpha(x)$ being some polynomials and k being the base field.

$$\begin{cases} \phi_1(x) = 0\,, \\ \cdots \\ \phi_n(x) = 0 \end{cases} \tag{2.2}$$

are the equations of M. Roughly speaking, $k[M]$ is the restriction of the ring of polynomials $k[x^1,\ldots,x^N]$ onto M having a nontrivial kernel — ideal $(\phi_1,\ldots,\phi_n)$. For instance, let X be the n-tuple point given by the equation $x^n = 0$ on the affine line $\mathbb{A}^1$; then for $n = 1$

$$k[M] \simeq k\,, \tag{2.3}$$

and for $n = 2$

$$k[M] \simeq \left\{a + b\epsilon \,|\, a, b \in k,\ \epsilon^2 = 0\right\}. \tag{2.4}$$

So algebraically a point can be easily distinguished from a double point which would never be possible in topological approach to the set of solutions of (2.2).

We would like to define the ring of differential operators on a subvariety. It is natural to use the construction of the ring of functions on the subvariety in terms of some resolvent. Namely, let $(K^\cdot, d_{\mathrm{K}})$ be a resolution of the structure sheaf $k[M]$ of the subvariety M in the affine space $\mathbb{A}^N$ with $K^p \simeq k[x] \otimes V_{n_p}$, $V_{n_p} \simeq k^{n_p}$, $\mathrm{gr} d_{\mathrm{K}} = -1$, and $H^0(K^\cdot) \simeq k[M]$. We define differential operators $\mathcal{D}_M$ on the algebraic variety M as the cohomologies $H_\delta^{0,0} H_d$ of the double sequence $\tilde{K}^{\cdot\cdot}$:

$$\tilde{K}^{p,q} \simeq \mathcal{D} \otimes \mathrm{Hom}(V_{n_q}, V_{n_p}) \simeq V_{n_p} \otimes \mathcal{D} \otimes V_{n_q}^*\,, \tag{2.5}$$

where $\mathcal{D}$ stands for the ring of differential operators on $\mathbb{A}^N$. The horizontal differential acts according to

$$dm^{p,q} = d_{\mathrm{K}} \circ m^{p,q} \in \tilde{K}^{p-1,q}\,, \tag{2.6}$$

and the vertical one

$$\delta m^{p,q} = m^{p,q} \circ d_{\mathrm{K}} \in \tilde{K}^{p,q+1} \tag{2.7}$$

for $m^{p,q} \in K^{p,q}$. Obviously, d and δ commute. The differential d_{K} of the resolvent $K^\cdot$ is the multiplication by a matrix whose entries are some functions, and we regard those functions as the zeroth order differential operators. Thereby one needs to consider some matrix differential operators as elements of $\tilde{K}^{\cdot\cdot}$ in order to define differential operators on a subvariety (see (2.5): $\mathrm{Hom}(V_{n_q}, V_{n_p}) \simeq \mathrm{Mat}(n_p, n_q)$).

Note that thus defined differential operators on a subvariety naturally act on the ring of functions on this subvariety. It is obvious from the following general consideration. The double complex just considered is a particular case of the general construction (e.g. [10]). Let (K, d_{K}) be a differential module. These data determine the bicomplex $(\tilde{K} = \mathrm{End}(K), d, \delta)$ with the two differentials

$$d : \mathrm{End}(K) \ni m \to d_{\mathrm{K}} \circ m, \quad \delta : m \to m \circ d_{\mathrm{K}} \tag{2.8}$$

(note that $d_{\mathrm{K}} \in \mathrm{End}(K)$). In the ultimately general case there exists the natural action of $H_\delta H_d(\tilde{K})$ on $H(K)$ described as follows. Take a cohomology class $[m] \in H_\delta H_d(\tilde{K})$; its representative m satisfies

$$\begin{aligned} dm &= 0 \iff d_{\mathrm{K}} \circ m = 0\,, \\ \delta m &= 0 \iff m \circ d_{\mathrm{K}} = d_{\mathrm{K}} \circ \tilde{m} \text{ for some } \tilde{m} \end{aligned} \tag{2.9}$$

and is determined up to

$$m \to m + d_{\mathrm{K}} \circ m_1 + m_2 \circ d_{\mathrm{K}}, \text{ where } d_{\mathrm{K}} \circ m_2 = 0\,. \tag{2.10}$$

Given $[f + d_{\mathrm{K}} g] \in H(K)$, the natural action

$$[m][f] = [mf] \tag{2.11}$$

is defined unambiguously (that is why $\mathcal{D}_M$ *does* act correctly on $k[M]$). This fact is established by the direct calculation. The key feature of the resolvent is its acyclicity everywhere but in one degree of grading, so the same should apply to the complex just considered: $(\mathrm{End}(K), d, \delta)$ also proves to be a resolvent with the nontrivial cohomologies $H \simeq \mathrm{End}(K)$ (the injectivity of the just described canonical homomorphism

$\varphi : H(\mathrm{End}(K)) \to \mathrm{End}(H(K))$ is proved in Appendix). Note that all the homomorphisms (endomorphisms) are understood as those over the base field k, $\mathrm{Hom}(\cdot,\cdot) = \mathrm{Hom}_k(\cdot,\cdot)$.

In the above considerations we replace $\mathrm{End}(k[x]) \to \mathcal{D}$ actually extracting a subclass $\mathcal{D}$ of all endomorphisms $\mathrm{End}(k[x])$ and call $H_\delta H_d(\tilde{K}^{\cdot\cdot})$ differential operators on the subvariety. It is completely in the spirit of algebraic geometry — recall the definition (2.1) of polynomial functions on an affine variety. Though the reduction procedure has the most natural description in terms of all endomorphisms $\mathrm{End}(k[x])$, i.e. all *integral* operators, local differential operators are of especial physical importance: these are local differential operators that describe massless modes (see further examples).

Now assume that constrains ϕ_α are regular, i.e. ϕ_α does not divide zero in $k[x]/(\phi_1,\ldots,\phi_{\alpha-1})$. Consider the Koszul resolvent $(K^\cdot, d_{\mathrm{K}})$ relevant to this case [11]:

$$\Lambda^n V \otimes k[x] \stackrel{d_{\mathrm{Koszul}}}{\longrightarrow} \Lambda^{n-1} V \otimes k[x] \stackrel{d_{\mathrm{Koszul}}}{\longrightarrow} \cdots \stackrel{d_{\mathrm{Koszul}}}{\longrightarrow} k[x] \longrightarrow 0\,; \tag{2.12}$$

where $V = \mathrm{span}\{\mathrm{e}_1,\ldots,\mathrm{e}_n\}$ is the n-dimensional vector space, $v = (\phi_1,\ldots,\phi_n)$; and the differential $d_{\mathrm{Koszul}} = i_v$ is given by the interior differentiation, $i_v \mathrm{e}_\alpha = \phi_\alpha$. The reduction procedure is described by the double complex $\tilde{K}^{\cdot\cdot}$:

$$\begin{array}{ccccccc}
\Lambda^n V \otimes \mathcal{D} \otimes \Lambda^n V^* & \stackrel{d}{\longrightarrow} & \Lambda^{n-1} V \otimes \mathcal{D} \otimes \Lambda^n V^* & \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} & \mathcal{D} \otimes \Lambda^n V^* & \longrightarrow 0 \\
\big\downarrow \delta & & \big\downarrow \delta & & \big\downarrow \delta & \\
\Lambda^n V \otimes \mathcal{D} \otimes \Lambda^{n-1} V^* & \stackrel{d}{\longrightarrow} & \Lambda^{n-1} V \otimes \mathcal{D} \otimes \Lambda^{n-1} V^* & \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} & \mathcal{D} \otimes \Lambda^{n-1} V^* & \longrightarrow 0 \\
\big\downarrow \delta & & \big\downarrow \delta & & \big\downarrow \delta & \\
\vdots & & \vdots & & \vdots & \\
\big\downarrow \delta & & \big\downarrow \delta & & \big\downarrow \delta & \\
\Lambda^n V \otimes \mathcal{D} & \stackrel{d}{\longrightarrow} & \Lambda^{n-1} V \otimes \mathcal{D} & \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} & \mathcal{D} & \longrightarrow 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & \\
0 & & 0 & & 0 &
\end{array} \tag{2.13}$$

The horizontal and vertical differentials d and δ are Koszul differentials, d acting on V and δ on V^* (to make notations for δ more convenient, we have replaced $\Lambda^k V^*$ with its Hodge dual $\Lambda^{n-k} V^*$). $\mathcal{D}$ is a free right or left module over $k[x]$ (but it is not a free bimodule!); so $H_d^{\cdot\cdot}(\tilde{K}^{\cdot\cdot})$ is concentrated in the last column, and $H_\delta^{\cdot\cdot}(\tilde{K}^{\cdot\cdot})$ is concentrated in the last row. These are the sufficient conditions that provide [12]

$$H_d H_\delta(\tilde{K}^{\cdot\cdot}) \simeq H_D(A^\cdot) \simeq H_\delta H_d(\tilde{K}^{\cdot\cdot})\,. \tag{2.14}$$

That is why $\mathcal{D}_M$ does not depend on whether we work with the right or left ideal (cf. [1]). The total complex is obtained via the contraction of the grading:

$$\mathrm{tot}\tilde{K}^{\cdot\cdot} \simeq A^\cdot \simeq \bigoplus_k A^k, \quad A^k = \bigoplus_{p+q=k} \tilde{K}^{p,q}\,. \tag{2.15}$$

Its differential

$$D = d + (-)^p \delta\,. \tag{2.16}$$

Let us stress that this definition of $\mathcal{D}_M$ is equivalent to that of [1]:[1]

$$H_d^{0,n} \simeq \frac{\mathcal{D}}{\sum\limits_\alpha \phi_\alpha \mathcal{D}}\,, \tag{2.17}$$

and the ring of differential operators on the subvariety is constructed as

$$H_\delta^{0,n} H_d \simeq \mathcal{D}_M \simeq \mathcal{N}/\mathcal{I}\,, \tag{2.18}$$

where the right ideal

$$\mathcal{I} = \left\{ \sum_\alpha \phi_\alpha \mathcal{D} \right\}. \tag{2.19}$$

The associated normalizer

$$\mathcal{N} = \{ m \in \mathcal{D} \,|\, m\phi_\alpha \subset \mathcal{I}\, \forall \alpha \} \tag{2.20}$$

is actually the maximal subalgebra in which $\mathcal{I}$ is a two-sided ideal. Introduction of normalizer makes the induced multiplication in $\mathcal{D}_M$ correctly defined. Such a prescription is precisely the quantum analog of the hamiltonian reduction [8].

Consider the codimension one example, i.e. a principal ideal $\mathcal{I}$ (e.g. $\mathcal{I} = (x^n)$). Contraction of the grading yields the total complex $A^\cdot$ (physicists usually call it BRST):

$$\begin{aligned}
&0 \longrightarrow \mathcal{D} \xrightarrow{d_1} \mathcal{D} \otimes (k \oplus k) \xrightarrow{d_2} \mathcal{D} \longrightarrow 0\,, \\
&d_1 f = (x^n f, f x^n)\,, \qquad d_2(g_1, g_2) = g_1 x^n - x^n g_2\,, \\
&d_2 \circ d_1 = 0\,.
\end{aligned} \tag{2.21}$$

What we need is $\ker d_2 / \operatorname{im} d_1$;

$$\ker d_2 = \{ (g_1, g_2) \,|\, g_1 x^n = x^n g_2 \}\,, \tag{2.22}$$

so a representative of $\ker d_2$ is uniquely determined by $g_1 \in \mathcal{N}_{\mathrm{r}}$ (of course, the index $_{\mathrm{r}}$ means "right," whereas $_{\mathrm{l}}$ means "left").

$$\operatorname{im} d_1 = \{ (x^n f, f x^n) \,|\, f \in \mathcal{D} \}\,, \tag{2.23}$$

and, finally,

$$\ker d_2 / \operatorname{im} d_1 \simeq \mathcal{D}_M\,. \tag{2.24}$$

One can express a representative of any (linear) subspace in $\mathcal{D}$ modulo $\mathcal{I}$ as

$$h(x, \partial) = h_0(\partial) + x h_1(\partial) + \cdots + x^{n-1} h_{n-1}(\partial) \tag{2.25}$$

for some $h_i = h_i(\partial)$. For $h \in \mathcal{N}$ this sum must also satisfy the equation ("$\equiv$" means "equal modulo $\mathcal{I}$," and $\mathcal{I}$ stands for the ideal in the ring of functions or differential operators depending on context)

$$h x^n \equiv 0 \tag{2.26}$$

[1] In [1] Merkulov used deformation quantization and the Moyal $\star$-product. It leads to integral operators in general, but if we restrict ourselves to polynomials in p, the Moyal algebra is nothing but the algebra of differential operators $\mathcal{D}$ (see further comments).

or, equivalently (we denote $\partial = \pi$ for convenience),

$$\left\{ \sum_{i=0}^{l} \binom{n}{l-i} \frac{d^{n-l+i} h_i(\pi)}{d\pi^{n-l+i}} = 0 \,, \qquad l = 0, \dots, n-1 \,. \right. \tag{2.27}$$

We have used the commutation relation

$$f(\pi) x^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \frac{d^k f}{d\pi^k} \,. \tag{2.28}$$

The general solution is

$$h_i(\partial) = \sum_{k=0}^{n-1} a_{i,k} \partial^k + \sum_{k=0}^{i-1} \frac{1}{(i-k)!} \left[\sum_{j=0}^{i-k-1} (-1)^{j+1} \binom{i-k-1}{j} a_{k,n-i+k+j} \partial^{n+j} \right] \tag{2.29}$$

with n^2 arbitrary integration constants $a_{i,k}$, $0 \le i,k \le n-1$; thus $\dim \mathcal{D}_M = n^2$. The natural action of $\mathcal{D}_M$ on $k[M]$ provides the celebrated isomorphism $\mathcal{D}_M \simeq \mathrm{Mat}(n)$. We see that in the case of the n-point local differential operators prove to be a sufficient subclass of $\mathrm{End}(k[x])$ to provide the complete matrix algebra, i.e. all endomorphisms $\mathrm{End}(k[M])$.

To complete the picture it remains to determine the $\operatorname{coker} d_2 \simeq \frac{\mathcal{D}}{\mathcal{I}_{\mathrm{l}} \oplus \mathcal{I}_{\mathrm{r}}}$. The obvious relation

$$\left(\sum_{i=0}^{n-1} x^i h_i \right) x^n \equiv \sum_{l=0}^{n-1} \sum_{i=0}^{l} \binom{n}{l-i} x^l \frac{d^{n-l+i} h_i}{d\pi^{n-l+i}} \tag{2.30}$$

implies

$$\frac{\mathcal{D}}{\mathcal{I}_{\mathrm{l}} \oplus \mathcal{I}_{\mathrm{r}}} = 0 \,. \tag{2.31}$$

3. Examples

In this section we study some examples in order to compare the result to that expected from physics. We find a perfect agreement. In particular, it is clear how the non-abelian degrees of freedom appear on each of the intersecting stacks. The quantization procedure also manages to find the zero modes of the D-instanton. After that we examine the present technique versus the resolution of singularities, and find that the quantization knows about the resolution. This fact nicely illustrates the connection of differential operators with string theory.

3.1 Intersecting stacks

Consider the case of $\mathbb{A}^2$ and the unique constraint (generator of the ideal) $\phi = x^m y^n$. The calculation (see appendix B) reveals that of importance are elements like (in this subsection h's are defined by (2.25), (2.29))

$$\mathcal{N} \supset \mathrm{span}\{ h(y, \partial_y) x^m f(x, \partial_x) \} \,. \tag{3.1}$$

They form a closed subalgebra $\mathcal{A}_y \subset \mathcal{D}_M$:

$$(h(y,\partial_y) x^m f(x,\partial_x))\, (\tilde{h}(y,\partial_y) x^m \tilde{f}(x,\partial_x)) \equiv (h\tilde{h}) x^m (f x^m \tilde{f}) \,. \tag{3.2}$$

This subalgebra is localized on the stack along the x axis: all the representatives vanish elsewhere. There also exists a similar subalgebra

$$\mathcal{A}_x = \mathrm{span}\{h(x, \partial_x) y^n f(y, \partial_y)\}\,. \tag{3.3}$$

The localization of these subalgebrae can be illustrated with the help of the Meier-Vietoris sequence for $\mathcal{D}_M$ constructed as follows. The two ideals corresponding to the two stacks are

$$\mathcal{I}_1 = (x^m)\,, \qquad \mathcal{I}_2 = (y^n)\,. \tag{3.4}$$

The exact Meier-Vietoris sequence for $k[M]$

$$\begin{array}{ccccccc} 0 \longrightarrow & k[M_1 \cup M_2] & \stackrel{\text{embedding}}{\longrightarrow} & k[M_1] \oplus k[M_2] & \stackrel{\text{difference}}{\longrightarrow} & k[M_1 \cap M_2] & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & \frac{k[x]}{\mathcal{I}_1 \mathcal{I}_2} & & \frac{k[x]}{\mathcal{I}_1} \oplus \frac{k[x]}{\mathcal{I}_2} & & \frac{k[x]}{\mathcal{I}_1 \oplus \mathcal{I}_2} & \end{array} \tag{3.5}$$

induces the acyclic double complex for $\mathrm{End}(k[M])$

$$\begin{array}{ccccccc} \mathrm{Hom}(k[M_1 \cap M_2], k[M_1 \cup M_2]) & \xrightarrow{d} & \mathrm{Hom}(k[M_1 \cap M_2], k[M_1] \oplus k[M_2]) & \xrightarrow{d} & \mathrm{End}(k[M_1 \cap M_2]) & \to 0 \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\ \mathrm{Hom}(k[M_1] \oplus k[M_2], k[M_1 \cup M_2]) & \xrightarrow{d} & \mathrm{End}(k[M_1] \oplus k[M_2]) & \xrightarrow{d} & \mathrm{Hom}(k[M_1] \oplus k[M_2], k[M_1 \cap M_2]) & \to 0 \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\ \mathrm{End}(k[M_1 \cup M_2]) & \xrightarrow{d} & \mathrm{Hom}(k[M_1 \cup M_2], k[M_1] \oplus k[M_2]) & \xrightarrow{d} & \mathrm{Hom}(k[M_1 \cup M_2], k[M_1 \cap M_2]) & \to 0 \\ \downarrow & & \downarrow & & \downarrow & \\ 0 & & 0 & & 0 & \end{array} \tag{3.6}$$

with d and δ defined by (2.8).

Remark We must fulfil the condition

$$\mathcal{I}_1 \cap \mathcal{I}_2 = \mathcal{I}_1 \mathcal{I}_2 \tag{3.7}$$

for (3.5) to be exact. That is why such a sequence is of little use say for a double point with $\mathcal{I}_1 = (x) = \mathcal{I}_2$.

The short sequence (3.5) provides the resolvent

$$k[M_1] \oplus k[M_2] \longrightarrow k[M_1 \cap M_2] \longrightarrow 0 \tag{3.8}$$

with $H^1 \simeq k[M_1 \cup M_2]$, $H^2 \simeq 0$. The resolvent for $\mathrm{End}(k[M_1 \cup M_2])$ can be built using the general recipe (2.8):

$$\begin{array}{ccccc} \mathrm{Hom}(k[M_1 \cap M_2], k[M_1] \oplus k[M_2]) & \xrightarrow{d} & \mathrm{End}(k[M_1 \cap M_2]) & \longrightarrow 0 \\ \downarrow \delta & & \downarrow \delta & \\ \mathrm{End}(k[M_1] \oplus k[M_2]) & \xrightarrow{d} & \mathrm{Hom}(k[M_1] \oplus k[M_2], k[M_1 \cap M_2]) & \longrightarrow 0 \\ \downarrow & & \downarrow & \\ 0 & & 0 & \end{array} \tag{3.9}$$

Using (3.6), one easily computes the necessary cohomologies

$$\mathrm{End}(k[M_1 \cup M_2]) \simeq H_\delta^{2,1} H_d \simeq \frac{d\mathrm{Hom}(k[M_1]\oplus k[M_2], k[M_1\cup M_2])}{\delta d\mathrm{Hom}(k[M_1\cap M_2], k[M_1\cup M_2])} \simeq \frac{\mathrm{Hom}(k[M_1]\oplus k[M_2], k[M_1\cup M_2])}{\delta\mathrm{Hom}(k[M_1\cap M_2], k[M_1\cup M_2])}\,. \tag{3.10}$$

In the case of the finite order operators we have

$$\mathrm{End}(k[M_1 \cup M_2]) \simeq \frac{\mathrm{span}\{(h_x h_y + \mathcal{A}_x, \tilde{h}_y \tilde{h}_x + \mathcal{A}_y)\}}{\mathrm{span}\{(h_x h_y, h_x h_y)\}}\,; \tag{3.11}$$

and that is why we associate $\mathcal{A}_x$, $\mathcal{A}_y$ with the two stacks and the finite-dimensional part $\{h_x h_y\}$ with the intersection.

A very important property is

$$\mathcal{A}_x \mathcal{A}_y = 0 = \mathcal{A}_y \mathcal{A}_x\,. \tag{3.12}$$

It allows to identify $\mathcal{A}_x$, $\mathcal{A}_y$ with the non-abelian degrees of freedom living on the worldvolume of the corresponding stack of D-branes. In the case of intersecting stacks the following excitation modes appear [3, 13, 14]:

$$\begin{pmatrix} \mathrm{Mat}(m) & \Psi \\ \Psi & \mathrm{Mat}(n) \end{pmatrix}. \tag{3.13}$$

Orthogonal subalgebrae $\mathrm{Mat}(m)$, $\mathrm{Mat}(n)$ are identified with $\mathcal{A}_x$, $\mathcal{A}_y$. Ψ corresponds to massive strings stretching between the two stacks and may be interpreted in terms of the non-local operators which appear if we omit the locality condition in our construction.[2]

If the two stacks are intersecting at an arbitrary non-zero angle, the result does not depend on its value. It also agrees well with the answer known from physics: the structure of massless modes does not depend on the non-zero angle between branes.

For an arbitrary number of stacks intersecting at a point the generator is given by $\phi = x^m y^n \prod_{i=3}^r \phi_i$ with $\phi_1 = x^m$, $\phi_2 = y^n$, $\phi_i = (\alpha_i x + y)^{\beta_i}$. The straightforward generalization is to consider

$$\mathcal{A}_x = \mathrm{span}\left\{ \left(y^n \prod_{i=3}^{r} \phi_i \right) h(x, \partial_x) g(y, \partial_y) \right\}. \tag{3.14}$$

It is a closed subalgebra localized on the first stack. This fact can be proved, for example, using (B.16). Just as in the case of the two stacks, the orthogonality relation (3.12) is satisfied for any pair of stacks. Denote

$$\mathcal{A}_{ij} = \left\{ \phi_1 \dots \phi_{\hat{i}} \dots \phi_{\hat{j}} \dots \phi_r H_{ij}(x, y, \partial) \right\}, \tag{3.15}$$

where H_{ij} satisfy

$$H_{ij} \phi_i \phi_j = \phi_i \phi_j \tilde{H}_{ij} \tag{3.16}$$

for some $\tilde{H}_{ij}(x, y, \partial) \in \mathcal{D}$. Obviously, $\mathcal{A}_i \subset \mathcal{A}_{ij}$ ($\mathcal{A}_i$ was defined in (3.14)). Some orthogonality relations hold for $\mathcal{A}_{ij}$, e.g.

$$\mathcal{A}_{12}\mathcal{A}_{34} = \mathrm{span}\left\{ (\phi_3 \phi_4 \phi_5 \dots \phi_r H_{12})(\phi_1 \phi_2 \phi_5 \dots \phi_r H_{34}) \right\} = 0\,. \tag{3.17}$$

So one can associate $\mathcal{A}_{ij}$ with the pair of stacks (i, j).

[2]Of course, our results apply to the non-supersymmetric (purely bosonic) theory.

3.2 A line with a double point

The constraints are

$$\begin{cases} xy = 0\,, \\ y^2 = 0\,, \end{cases} \tag{3.18}$$

determining the ring of functions

$$k[M] \simeq \big\{ f(x) + A\epsilon \,|\, \epsilon^2 = 0 \big\}\,. \tag{3.19}$$

The subring $\{f(x)\}$ is the ring of functions on the affine line, and the nilpotent is responsible for the D-instanton located at the origin.

Let us determine $\mathcal{D}_M$. Any operator modulo $\mathcal{I}$ can be brought to the form

$$h = \sum_n x^n f_n(\pi_x, \pi_y) + y g(\pi_x, \pi_y)\,. \tag{3.20}$$

For convenience we define $\pi_x = \partial_x$, $\pi_y = \partial_y$. Belonging of h to $\mathcal{N}$ requires

$$\begin{cases} \dfrac{\partial^2 f_n}{\partial \pi_y^2} = 0\,, \qquad n = 0, 1, \ldots\,, \\ \dfrac{\partial^2 g}{\partial \pi_y^2} + 2\dfrac{\partial f_0}{\partial \pi_y} = 0\,, \\ \dfrac{\partial f_{n-1}}{\partial \pi_y} + \dfrac{\partial^2 f_n}{\partial \pi_x \partial \pi_y} = 0\,, \qquad n = 1, 2, \ldots\,, \\ \dfrac{\partial^2 f_0}{\partial \pi_x \partial \pi_y} = 0\,, \\ \dfrac{\partial^2 g}{\partial \pi_x \partial \pi_y} + \dfrac{\partial f_0}{\partial \pi_x} = 0\,. \end{cases} \tag{3.21}$$

The general solution in the class of finite order operators is

$$h = f(x, \partial_x)(1 - y\partial_y) + y g(\partial_x) + C y \partial_y\,. \tag{3.22}$$

Obviously, $\{f(x, \partial_x)(1 - y\partial_y)\}$ is the subring of differential operators on the line, which can be established through its action on $k[M]$:

$$\begin{aligned} f(x, \partial_x)(1 - y\partial_y)\,\phi(x) \,&=\, f(x, \partial_x)\,\phi(x)\,, \\ f(x, \partial_x)(1 - y\partial_y)\,y \,&=\, 0\,. \end{aligned} \tag{3.23}$$

It describes the original D-brane's degrees of freedom. Analogously, $y\partial_y$ is related to the D-instanton,

$$\begin{aligned} y\partial_y\,\phi(x) \,&=\, 0\,, \\ y\partial_y\, y \,&=\, y\,, \\ (y\partial_y) \circ (y\partial_y) \,&=\, y\partial_y\,. \end{aligned} \tag{3.24}$$

At last, zero modes of the D-instanton are the physical assignment of $\{y g(\partial_x)\}$.

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3.3 A point on the cusp

The cusp is defined by

$$y^2 - x^3 = 0\,. \tag{3.25}$$

Additional equation $x = a \neq 0$ sets two different points on this curve. The related algebra is the matrix algebra Mat(2), off-diagonal elements being represented by non-local operators (shift operators). When $a \to 0$ these two points glue together to form a double point with the ring of functions (2.4) and the related algebra of differential operators $\mathcal{D}_M \simeq \mathrm{Mat}(2)$.

Let us resolve the singularity. This goal is reached blowing up the origin, i.e. saying that the good coordinate is $s = y/x$ instead of y; then the equation of the curve (3.25) takes the form

$$x^2(x - s^2) = 0\,. \tag{3.26}$$

The two exceptional lines $x^2 = 0$ are discarded, and finally the curve becomes the quadratic parabola $x = s^2$ after the resolution. Combining this equation with the equation $x = a$, we again arrive to the double point,

$$\begin{cases} s^2 = 0\,, \\ x = 0\,, \end{cases} \tag{3.27}$$

in the $a \to 0$ limit.

It is not difficult to explain why it happens so. As far as $a \neq 0$ the blow-up is a good diffeomorphism in a neighbourhood of the point $(x = a,\ y = \sqrt{a^3})$, and the smooth $a \to 0$ limit for $k[M]$ and $\mathcal{D}_M$ ("smooth" means the dimension, not the structure constants!) is not surprising for algebraic geometry — that is why one might expect the coincidence of the two results for $\mathcal{D}_M$.

The similar scenario takes place in the case of orbifolds considered below.

3.4 A point on the $\mathbb{C}^2/\mathbb{Z}_m$ orbifold

Given the action of $\mathbb{Z}_m$ on $\mathbb{C}^2$ as

$$\mathbb{C}^2 \ni (u, v) \to (\epsilon u, \epsilon^{-1} v) \quad \text{with} \quad \epsilon = \sqrt[m]{1}\,, \tag{3.28}$$

one has three invariants $x = u^m$, $y = v^m$, $z = uv$ satisfying

$$z^m = xy\,; \tag{3.29}$$

this way the $\mathbb{C}^2/\mathbb{Z}_m$ orbifold is embedded into $\mathbb{C}^3$. Here we dwell on the $\mathbb{Z}_2$ orbifold and carry the more general case out into Appendix.

Consider first a quadruple-point given by the set of equations

$$\begin{cases} x = 1\,, \\ y^2 = 0\,, \\ z^2 - xy = 0\,. \end{cases} \tag{3.30}$$

The multiplicity is defined by splitting these coincident points, e.g. via a small deformation of the r.h.s. The element of the quotient ring of differential operators $\mathcal{D}_M$ is represented as

$$h \equiv \sum_{i=1}^{16} C^i \mathrm{e}_i \tag{3.31}$$

with the base vectors e_i given by

$$\begin{aligned}
\mathrm{e}_1 &= 1\,, \quad \mathrm{e}_2 = y\,, \quad \mathrm{e}_3 = z\,, \quad \mathrm{e}_4 = yz\,, \quad \mathrm{e}_5 = zy\partial_z\,, \quad \mathrm{e}_6 = y\partial_z - zy\partial_z^2\,,\\
\mathrm{e}_7 &= \frac{1}{2}yz\partial_z^2 + yz\partial_y\,, \quad \mathrm{e}_8 = \frac{1}{6}yz\partial_z^3 + yz\partial_y\partial_z\,, \quad \mathrm{e}_9 = -y\partial_z^2 + z\partial_z + \frac{2}{3}yz\partial_z^3\,,\\
\mathrm{e}_{10} &= \frac{1}{6}y\partial_z^3 + y\partial_y\partial_z - \frac{1}{6}yz\partial_z^4 - yz\partial_y\partial_z^2\,, \quad \mathrm{e}_{11} = \partial_z + \frac{2}{3}y\partial_z^3 - z\partial_z^2 - \frac{1}{3}yz\partial_z^4\,,\\
\mathrm{e}_{12} &= -\frac{1}{3}y\partial_z^3 + \frac{1}{2}z\partial_z^2 + \frac{1}{12}yz\partial_z^4 + z\partial_y - yz\partial_y^2 - yz\partial_y\partial_z^2\,, \quad \mathrm{e}_{13} = \frac{1}{2}y\partial_z^2 + y\partial_y - \frac{1}{3}yz\partial_z^3\,,\\
\mathrm{e}_{14} &= \frac{1}{2}\partial_z^2 + \frac{1}{12}y\partial_z^4 + \partial_y - y\partial_y^2 - y\partial_y\partial_z^2 - \frac{1}{3}z\partial_z^3 + \frac{2}{3}yz\partial_y\partial_z^3\,,\\
\mathrm{e}_{15} &= -\frac{1}{6}y\partial_z^4 - y\partial_y\partial_z^2 + \frac{1}{6}z\partial_z^3 + \frac{1}{12}yz\partial_z^5 + z\partial_y\partial_z - yz\partial_y^2\partial_z + \frac{1}{3}yz\partial_y\partial_z^3\,,\\
\mathrm{e}_{16} &= \frac{1}{6}\partial_z^3 + \frac{1}{12}y\partial_z^5 + \partial_y\partial_z - y\partial_y^2\partial_z + \frac{1}{3}y\partial_y\partial_z^3 - \frac{1}{6}z\partial_z^4 - \frac{1}{36}yz\partial_z^6 - z\partial_y\partial_z^2 + yz\partial_y^2\partial_z^2\,.
\end{aligned}$$

The ring $\mathcal{D}_M$ is isomorphic to $\mathrm{Mat}(4)$. To derive this representation, the action of $\mathcal{D}_M$ on $k[M]$ can be used again:

$$\sum C^i \mathrm{e}_i \longleftrightarrow \begin{pmatrix} C_1 & C_{14} & C_{11} & C_{16} \\ C_2 & C_1 + C_{13} & C_3 + C_6 & C_{10} + C_{11} + C_{12} \\ C_3 & C_{12} & C_1 + C_9 & C_{14} + C_{15} \\ C_4 & C_3 + C_7 & C_2 + C_5 & C_1 + C_8 + C_9 + C_{13} \end{pmatrix} \tag{3.32}$$

Thus the system (3.30) really defines four coincident points; and if we are going to consider a single point, the appropriate constraints are

$$\begin{cases} y = a \neq 0\,, \\ z = b\,, \\ z^2 - xy = 0\,. \end{cases} \tag{3.33}$$

In this case $\mathcal{D}_M \simeq \mathbb{C}$ just as one expects for a single point. What happens as the point approaches the singularity, i.e. $a \to 0$? The only way to set a point sitting at the singularity is to put some restrictions on both x and y, e.g.

$$\begin{cases} x = 0\,, \\ y = 0\,, \\ z^2 - xy = 0\,. \end{cases} \tag{3.34}$$

Now one reads the resulting algebra as

$$\mathcal{D}_M \simeq \mathrm{Mat}(2)\,. \tag{3.35}$$

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The geometric origin of this phenomenon is clear: each singular point behaves as $m = 2$ regular points glued together causing the jump of $\dim \mathcal{D}_M$ (the phenomenon is observed for any m — see about this in appendix C).

The same result is obtained via the resolution technique. The birational map is

$$\begin{cases} y = vx\,, \\ z = wx\,. \end{cases} \tag{3.36}$$

Next we deform (3.34) substituting $x = a \to 0$ for $x = 0$. After the blow-up

$$\begin{cases} x = a \to 0\,, \\ y = 0\,, \\ z^2 = xy\,. \end{cases} \to \begin{cases} x = a \to 0\,, \\ av = 0\,, \\ w^2 = v\,. \end{cases} \to \begin{cases} x = a \to 0\,, \\ v = 0\,, \\ w^2 = 0\,. \end{cases} \tag{3.37}$$

The latter is the proper equivalent set of equations generating the same ideal and enjoying the smooth $a \to 0$ limit that yields $\dim \mathcal{D}_M = 4$.

The ultimate question is why do we need to deform $x = 0 \to x = a$? Such a behaviour is typical for blow-ups and is used for finding what points of the exceptional line belong to our curve. To find whether a point belongs to the curve, we move the point along the fiber of the tautological bundle and check whether the moved point is far from the curve. (Recall that geometrically the projective plane $\mathbb{P}^2$ that replaces the origin of $\mathbb{C}^3$ after the blow-up is called the plane of directions.)

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A. Proof of injectivity of the canonical homomorphism

We are going to verify that the kernel of the canonical homomorphism $\varphi : H(\mathrm{End}(K)) \to \mathrm{End}(H(K))$ is trivial. Let $[m] \in H_\delta H_d \tilde{K}$; its representative is chosen up to

$$m \to m' = m + n \circ d_{\mathrm{K}} + d_{\mathrm{K}} \circ n', \quad \text{where} \quad d_{\mathrm{K}} \circ n = 0\,. \tag{A.1}$$

The original differential module K decomposes as

$$K = \frac{K}{Z} \oplus Z\,, \tag{A.2}$$

where $Z = \ker d_{\mathrm{K}}$. The reason for such a decomposition is that in the case of interest the module K is actually a vector space over the base field k. Note that the image $\operatorname{im} m \subset Z$ since $d[m] = [0]$.

The $n \circ d_{\mathrm{K}}$ term does not affect the action of m on Z; let us investigate its action on $C \in K/Z : mC + n \circ d_{\mathrm{K}} C \in Z$. Picking up an appropriate $n : B \to Z$ ($B = \operatorname{im} d_{\mathrm{K}}$ — coboundaries), we achieve

$$(m + n \circ d_{\mathrm{K}})C = 0 \;\forall C \in \frac{K}{Z}\,. \tag{A.3}$$

We have used the projectivity and the isomorphism

$$\frac{K}{Z} \simeq d_{\mathrm{K}} \frac{K}{Z} \simeq B\,. \tag{A.4}$$

After such a choice of the representative $[m] \in \ker \varphi$ implies $\operatorname{im} m \subset B$ requiring $m = d_{\mathrm{K}} \circ n'$, so $[m] = [0]$ and

$$\ker \varphi = [0]\,. \tag{A.5}$$

B. Calculation of $\mathcal{D}_M$ for intersecting stacks

B.1 Orthogonal stacks

The right ideal and normalizer are defined as above:

$$\begin{aligned}
\mathcal{I} &= x^m y^n \mathcal{D}\,,\\
\mathcal{N} &= \{f(x,y,\partial) \,|\, f x^m y^n = x^m y^n g \text{ for some } g \in \mathcal{D}\}\,.
\end{aligned} \tag{B.1}$$

Next we proceed to determine the moduli of the quotient $\mathcal{D}_M \simeq \mathcal{N}/\mathcal{I}$. In this case we can choose a representative of $\mathcal{N}$ as

$$f \equiv \sum_{k<m \text{ or } l<n} x^k y^l f_{kl}(\partial_x, \partial_y)\,. \tag{B.2}$$

The condition $f \in \mathcal{N}$ after the repeated use of the commutation relations (2.28) yields

$$0 = \sum_{l,k} \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} x^{m+l-i} y^{n+k-j} \frac{\partial^{i+j} f_{lk}}{\partial \pi_x^i \partial \pi_y^j}\,, \tag{B.3}$$

which is equivalent to

$$\partial_{\pi_x}^m \partial_{\pi_y}^n f_{\alpha\beta} = -\underbrace{\sum_{l=0}^{\min(m,\alpha)} \sum_{k=0}^{\min(n,\beta)}}_{k+l>0} \binom{m}{l} \binom{n}{k} \frac{\partial^{m+n-l-k}}{\partial \pi_x^{m-l} \partial \pi_y^{n-k}} f_{\alpha-l,\beta-k}\,, \tag{B.4}$$

where $\alpha < m$ or $\beta < n$. So we obtain the following recurrent relations:

$$f_{\alpha\beta} = A^{(0)}_{\alpha\beta}(\pi_x) + \cdots + A^{(n-1)}_{\alpha\beta}(\pi_x)\pi_y^{n-1} + B^{(0)}_{\alpha\beta}(\pi_y) + \cdots + B^{(m-1)}_{\alpha\beta}(\pi_y)\pi_x^{m-1} -$$
$$- \underbrace{\sum_{l=0}^{\min(m,\alpha)} \sum_{k=0}^{\min(n,\beta)}}_{k+l>0} \binom{m}{l}\binom{n}{k} \int^{\pi_x} d\pi_x^{(1)} \ldots \int^{\pi_x^{(l-1)}} d\pi_x^{(l)} \int^{\pi_y} d\pi_y^{(1)} \times \cdots$$
$$\cdots \times \int^{\pi_y^{(k-1)}} d\pi_y^{(k)} f_{\alpha-l,\beta-k}(\pi_x^{(l)}, \pi_y^{(k)}) \,. \tag{B.5}$$

For example, in the case of $m = 1$ and $n = 1$

$$\begin{aligned} f_{0,0} &= A_0(\pi_x) + B_0(\pi_y)\,, \\ &\cdots \\ f_{k,0} &= -\int d\pi_x\, f_{k-1,0}(\pi_x, \pi_y) + A_k(\pi_x) + B_k(\pi_y)\,, \\ &\cdots \end{aligned} \tag{B.6}$$

and the similar expressions are implemented for $f_{0,k}$.

The parameterization of the representative being fixed, a point of the moduli space of $\mathcal{D}_M$ proves to be the infinite number of some functions; and it is not trivial to bring the algebra to a simpler form. There arose such a complication because we had actually dealt with some integral operators (differential operators of infinite order) in this calculation. To understand this better, consider two distinct points in $\mathbb{A}^1$,

$$x^2 - 1 = 0\,. \tag{B.7}$$

Then the quotient algebra

$$\mathcal{D}_M = \left\{ A + Bx + C(x-1)\mathrm{e}^{2\partial_x} + D(x+1)\mathrm{e}^{-2\partial_x} \right\} \simeq \mathrm{gl}(2) \tag{B.8}$$

is represented in terms of shift operators [8]. Should we restrict ourselves to the finite order operators, only the diagonal matrices would survive. The finiteness of the order is a kind of the locality condition.

What are the finite order solutions of (B.4)? In the algebraic situation some maximal α_0 must exist, so that $f_{\alpha\beta} = 0$ for $\alpha > \alpha_0$. It allows us to set $\alpha = \alpha_0 + m \geq m$ in (B.4). Only terms with $l = m$ survive so

$$\sum_{k=0}^{\beta} \binom{n}{k} \frac{\partial^{n-k} f_{\alpha_0,\beta-k}}{\partial \pi_y^{n-k}} = 0\,, \qquad \beta = 0, \ldots, n-1\,; \tag{B.9}$$

these equations coincide with (2.27) written w.r.t. π_y for $f_{\alpha_0,\beta}$. Now let $\alpha = \alpha_0 + m - 1$. Taking into account (B.9) for $f_{\alpha_0,\beta}$, we obtain the same equations for $f_{\alpha_0-1,\beta}$; then for $f_{\alpha_0-2,\beta}, \ldots$ Running over the same procedure for $\beta = \beta_0$, $\beta_0 - 1$, …, one establishes that

$$\mathcal{D}_M \simeq \mathrm{span}\left\{ h_y(x,\partial_x) y^n f_y(y,\partial_y) + h_x(y,\partial_y) x^m f_x(x,\partial_x) + \tilde{h}_y(x,\partial_x)\tilde{h}_x(y,\partial_y) \right\}, \tag{B.10}$$

where h's satisfy (2.27) and are thereby given by (2.25), (2.29). Note that they are in the one-to-one correspondence with the $m \times m$- (correspondingly $n \times n$-) matrices. In the holomorphic case one can allow f_i to be a power series in x^i $(i = x, y)$.

B.2 Two stacks intersecting at an arbitrary angle

$$\phi = x^m(\alpha x + y)^n . \tag{B.11}$$

Performing the linear change of variables

$$\begin{cases} x' = x \,, \\ y' = \alpha x + y \,, \end{cases} \qquad \begin{cases} \partial'_x = \partial_x - \alpha \partial_y \,, \\ \partial'_y = \partial_y \,, \end{cases} \tag{B.12}$$

one finds that the algebra $\mathcal{D}_M$ depends on no angular parameter α. In fact, in the present construction one needs no metric; so all the transverse directions to the stack are of equal importance. The isomorphism $\mathcal{D}_M(\alpha) \simeq \mathcal{D}_M(\alpha = 0)$ acts on the representatives as

$$f^{(\alpha=0)}(x, y, \partial_x, \partial_y) \longleftrightarrow f^{(\alpha)}(x, y, \partial_x, \partial_y) = f^{(\alpha=0)}(x, \alpha x + y, \partial_x - \alpha\partial_y, \partial_y) . \tag{B.13}$$

It will help us to derive the following technical statement important in the future. Consider again m coincident lines in $\mathbb{A}^2$ with $\phi = x^m$. One can apply some speculations concerning transversal coordinates: $\mathcal{N}$ modulo $\mathcal{I}$ is

$$\mathcal{N} = \operatorname{span}\{g(y, \partial_y)h(x, \partial_x)\} + \mathcal{I} . \tag{B.14}$$

On making a change of coordinates $x \to x$, $y \to \alpha x + y$, $\partial_x \to \partial_x - \alpha\partial_y$, $\partial_y \to \partial_y$, $\mathcal{N}/\mathcal{I}$ becomes

$$\mathcal{N} = \operatorname{span}\{g(\alpha x + y, \partial_y)h(x, \partial_x - \alpha\partial_y)\} + \mathcal{I} . \tag{B.15}$$

These two equations prove that for any $h(x, \partial_x)$, $g(y, \partial_y)$ there always exist some $\tilde{h}(x, \partial_x - \alpha\partial_y)$, $\tilde{g}(\alpha x + y, \partial_y)$ such that

$$gh \equiv \tilde{g}\tilde{h} . \tag{B.16}$$

C. More orbifolds

The set of equations defining the n-point at the singularity of the $\mathbb{Z}_m$ orbifold is

$$\begin{cases} x = 0 \,, \\ y^n = 0 \,, \\ z^m = xy \end{cases} \iff \begin{cases} x = 0 \,, \\ y^n = 0 \,, \\ z^m = 0 \,. \end{cases} \tag{C.1}$$

The blow-up is again performed using (3.36), and (C.1) takes the form

$$\begin{cases} x = a \,, \\ v^n a^n = 0 \,, \\ w^m a^m = 0 \end{cases} \to \begin{cases} x = a \to 0 \,, \\ v^n = 0 \,, \\ w^m = 0 \,. \end{cases} \tag{C.2}$$

We do not need to resolve the singularity completely: the unresolved part of the singularity is located on the line at infinity apart from the doings.

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